COMBUSTION IN NARROW CAVITIES

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Solid-propellant motors are occasionally subject to instabilities that lead to explosions. This is often because the propellant contains excessively large crack-type cavities, chiefly created during the manufacturing process. When the combustion front approaches the edge of such a cavity, as a result of the elevated pressure in the combustion chamber, combustion rapidly envelops the entire cavity. If the cavity is sufficiently narrow and long, as a result of the impeded gas flow the pressure in it reaches such values that the system becomes unstable. Depending on the type of propellant, the instability mechanism may take two quite different physical forms: a) local volume burning at the end of the cavity; b) local destruction of the propellant. A theoretical description of these effects is offered below*. In Sec.1 the problem is stated, and in Sec. 2 a local instability criterion is formulated. Then the solutions of plane stationary problems are considered, and a sufficient condition of stability of the system is found in analytic form for one very simple case (Sec. 3).

1. Formulation of the Problem.

Let a burning solid at time $t = 0$ constitute a half-space $x > 0$, at the boundary $x = 0$ of which a combustion reaction is taking place. Within the material in the plane $y = 0$, there is a crack-like cavity, the surface of which is also burning (see Fig. 1). The extension of the argument to include the case of a body and a crack of arbitrary shape does not present any fundamental difficulties. We assume that all the reactants are uniformly distributed in the solid phase, and that the reaction products are gaseous. We neglect the thickness of the reaction zone, assuming that the combustion front coincides with the surface of the body as a whole.

With a view to practical applications, we will confine ourselves to times small as compared with the characteristic stress relaxation time in the body and the characteristic conductive heating time. In this case the body may be assumed elastoplastic and its temperature constant. We will consider only the quasistatic deformation process, assuming that the characteristic times of the process are large as compared with the characteristic time of elastic wave propagation.

We write the equations of the theory of small deformations in the elastic region:

the equilibrium equations

$$\sigma_{ij,j} = 0 \qquad (i, j = 1, 2, 3) \tag{1.1}$$

Hooke's law for a homogeneous isotropic body

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\theta \qquad (\theta = \varepsilon_{ij}) \tag{1.2}$$

*O. I. Leipunskii and Z. V. Kirsanova have formulated and solved a specific plane problem of this type: "Mechanical stability of burning cracks in a propellant," Abstr. Proc. First All-Union Symposium on Combustion and Explosion [in Russian], Moscow (1968). As far as can be judged from this short account, the propellant is assumed to be an elasto-brittle material, and local volume burning at the end of the crack is not considered. The possibility of making these assumptions in relation to actual systems requires careful experimental verification (there is no mention of this in the abstract) and, in any event, seriously restricts the region of practical applications.

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the kinematic strain–displacement relation

$$\varepsilon_{ij} = {}^1/_2 (u_{i,j} + u_{j,i}) \tag{1.3}$$

Here, u_i, ε_{ij}, σ_{ij} are displacements, strains, and stresses, respectively; λ and μ are Lamé's constants; the subscripts 1, 2, and 3 correspond to x, y, and z (Fig. 1).

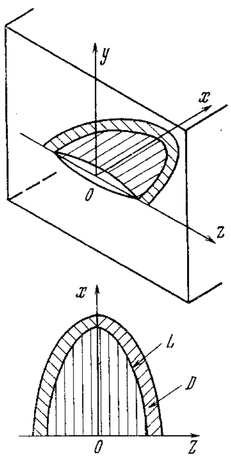

Fig. 1

We will consider only plastic effects in the neighborhood of the edge of the cavity. We employ the approximate Dugale model, according to which the plastic region is concentrated in a certain narrow region on the continuation of the crack; the size of this region must be determined from the solution of the problem. For simplicity, we confine ourselves to the case of symmetry about the plane y = 0. In this case the plastic strains are concentrated in the same plane in a certain neighborhood D of the crack contour in plan (Fig. 1), and

$$\sigma_y = \sigma_s, \quad \sigma_{xy} = \sigma_{zy} = 0 \qquad (y = 0, (x, z) \in D) \tag{1.4}$$

Here, σ_s is the yield point in tension (the actual $\sigma - \varepsilon$ diagram is approximated by the Prandtl diagram). Everywhere in what follows the boundary conditions are removed from the surface of the cavity in the region D to the plane y = 0, in exactly the same way as, for example, in slender wing theory.

The Dugdale model has received satisfactory experimental confirmation in two cases of practical importance: a) thin plates; b) composite propellants with a polymer matrix, when the adhesion strength is less than the strength of the polymer. In the latter case, strictly speaking, high-elastic rather than plastic strains are concentrated in region D. However, it is fairly obvious that this is unimportant within the framework of discontinuous solutions of the theory of small elastic deformations. We note that a composite rocket propellant usually consists of crystalline particles distributed in a polymer matrix.

We denote the thickness of the cavity, not known in advance, by 2h, the relative displacement of the opposite walls of the cavity by 2v, and the thickness of the burnt layer on one wall by h_c. Obviously, all these quantities are certain unknown functions of x, z, and t. They are related with each other by the following expression:

$$h = h_c + v \tag{1.5}$$

Equations of Flow of the Gas in the Cavity. We assume that the gas flow in the cavity is locally isentropic and irrotational and that the gas is an ideal gas. At any point x, y, z we have a closed system of equations

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \quad \frac{d\mathbf{v}}{dt} = -\frac{1}{\rho} \operatorname{grad} p, \quad \frac{d}{dt} \frac{p}{\rho^{\varkappa}} = 0 \tag{1.6}$$

Here, p, ρ, and v are the pressure, density, and velocity vector of the gas, respectively; $\varkappa$ is the adiabatic exponent.

If we simplify the general equations (1.6) by taking into account the condition $h \ll L$, where L is the characteristic linear dimension of the cavity, we encounter difficulties related to those that arise in the theory of elastic shells and the theory of turbulence. Since these difficulties are not reflected in the literature, it is worthwhile dwelling on the derivation of the basic equations of flow of the gas in the cavity.

Two exact methods are possible. The first is based on averaging operations applied to Eqs. (1.6) with the object of closing the system of equations for the mean values of the unknown functions:

$$\langle p \rangle = \frac{1}{2h} \int_{-h}^{+h} p \, dy, \quad \langle \mathbf{v} \rangle = \frac{1}{2h} \int_{-h}^{+h} \mathbf{v} \, dy, \ \ldots \tag{1.7}$$

and the higher-order moments. However, owing to the nonlinearity of Eqs. (1.6), this is not feasible (closure problem similar to that encountered in connection with turbulence). The second method is based on the introduction into the unknown functions of a physical small parameter (for example, h/L) and expansion of the functions in a series in y and the small parameter. In this case, for the successive terms of the expansion we obtain closed systems of equations, which in principle could be solved successively in order of increasing index. However, because the boundary conditions in boundary-value problems with different indices are interrelated, the resulting difficulties cannot be overcome by exact methods (not to mention the difficulties of summing the asymptotic series).

Thus, in closing the system of equations or formulating the boundary conditions, it is necessary to make certain physical assumptions that simplify the problem (Kirchhoff theory for elastic plates, the theories of Taylor and Prandtl in turbulence, etc.). Following the above-mentioned semiempirical method, we adopt the hypothesis of "plane sections":

$$p = p(x, z, t), \quad \rho = \rho(x, z, t), \quad v_x = v_x(x, z, t), \quad v_z = v_z(x, z, t) \tag{1.8}$$

Then from (1.6) for the components v_x and v_y we easily obtain the following system of equations (the averaging operation is applied to the continuity equation):

$$\begin{gathered} h\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(h\rho v_x) + \frac{\partial}{\partial z}(h\rho v_z) = \rho v_n \\ \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_z\frac{\partial v_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_z\frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \\ \frac{\partial}{\partial t}\left(\frac{p}{\rho^\varkappa}\right) + v_x\frac{\partial}{\partial x}\left(\frac{p}{\rho^\varkappa}\right) + v_z\frac{\partial}{\partial z}\left(\frac{p}{\rho^\varkappa}\right) = 0 \end{gathered} \tag{1.9}$$

Here v_n is the flow velocity of the gas along the normal to the cavity surface. We write the equation of conservation of mass at the combustion front

$$\rho_s\frac{\partial h_c}{\partial t} = \rho\left(v_n - \frac{\partial h_c}{\partial t}\right)$$

where ρ_s is the density of the solid phase. Since $\rho_s \gg \rho$,

$$\rho v_n = \rho_s\frac{\partial h_c}{\partial t} \tag{1.10}$$

We also write the equation of state of an ideal gas:

$$p = \rho g\frac{R_0}{m}T \qquad R_0 = 1.99 \text{ cal/mole·deg} \tag{1.11}$$

Here, m is the molecular weight of the gas, T the gas temperature, g the acceleration of gravity at a given point of space (the principal local characteristic of the gravitational field).

We take the burning rate in the form of a certain empirically determined relation

$$\frac{\partial h_c}{\partial t} = f(p)(1 + bv_x^2 + bv_z^2) \tag{1.12}$$

where b is an empirical constant (in the absence of erosion, b = 0), and $f(p)$ is a certain function.

The system of 23 equations (1.1)-(1.3), (1.5), (1.9)-(1.12) with the corresponding initial and boundary conditions is used to determine the 23 unknown functions.

2. Local Instability Criterion

Under certain critical conditions the rate of change of the linear dimensions of the cavity in plan may be much greater than the rate of change of the cavity thickness. This is because of the following physical factors: a) local destruction of the propellant at the end of the cavity as a result of stress concentration; b) a sharp increase in burning rate at the end of the cavity as a result of local disintegration of the material and an increase in the combustion surface (local volume burning). The latter factor is also controlled by the local strain concentration.

These physical local instability mechanisms, which play different roles in different materials, have a common origin — local strain concentration. This makes it possible to formulate the following limiting equilibrium criterion. Within the framework of the Dugdale model the mutual displacement of the opposite walls of the cavity on its contour L is always less than or equal to 2δ:

$$v(x,\ z,\ t) \leqslant \delta \qquad (x,\ z \in L) \tag{2.1}$$

Here δ is a certain constant of the burning propellant. From physical considerations, δ can depend only on the propellant temperature and gas pressure at the corresponding point of the contour L.

In accordance with (2.1), at $v < \delta$ the process is locally unstable, and the plan dimensions of the cavity do not vary (the growth of the cavity that takes place, in particular, as a result of stable burning is neglected, which is wholly permissible only at suffieiently small times, when $h \ll L$); as soon as $v = \delta$ at one point on the contour L, a locally unstable "combustion–disintegration" process begins in a small neighborhood of that point. It should be kept in mind that in certain cases the system as a whole may still remain stable (i.e., the change in the configuration of the cavity at points where $v = \delta$ is "not catastrophically rapid"). This case is perfectly realistic. Accordingly, the question of the stability of the system as a whole should be studied independently.

If the Dugdale model is not applicable, i.e., if the plastic strains are "spread" over a certain region, it is necessary to use a local criterion based on the stress intensity coefficient of the hyperfine structure proposed in [1].

When the fine structure of the end of the cavity is a meaningful concept, all these criteria are equivalent to the Irwin condition

$$K_I \leqslant K_{Ic} \qquad (K_{Ic}{}^2 = 2\sigma_s E\delta) \tag{2.2}$$

Here, K_I is the stress intensity coefficient, K_{Ic} is the strength. In practice, the most characteristic sign of the formation of a fine structure is the appearance of a scale effect.

Condition (2.1) was first proposed as a brittle fracture criterion for cracked bodies by M. Ya. Leonov and V. V. Panasyuk [2]; in recent years it has been widely accepted abroad (without reference to Soviet authors). In connection with the present problem, the criterion acquires a new physical significance.

In order to determine the constants δ and K_{Ic}, the following experimental scheme is the most convenient. A through notch, simulating a defect, is formed in a thin slab of the material investigated. The slab should be thick enough for δ and K_{Ic} not to depend on the slab thickness. For the same reason, the radius of the notch should not be too large. These are practical limits peculiar to each material; as far as composite propellants are concerned they are not burdensome. Then the notched slab is subjected to loading and ignited in a chamber at a suitable pressure. The loading schemes may vary (the simplest is central bending). The proposed method is convenient in that, if the slab is thin enough, the pressure in the cavity will be the same as in the chamber. In this case, the quantities v and K_I can be computed relatively easily in the form of certain functions of the external load and the geometry of the slab together with the material constants. Having measured the limiting load, it is easy to determine δ and K_{Ic}.

3. The Plane Stationary Problem

We will consider the important case of the plane stationary problem, when in the general system of equations presented above we can substitute

$$v_z = 0, \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial \rho}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial v_x}{\partial t} = 0 \tag{3.1}$$

In the xy plane the cavity is a cut of length l along the x axis; the thickness of the cut h is much less than l. On the continuation of the cut there is a plastic layer of zero thickness and length d to be determined.

In the given case, neglecting erosion, we reduce the system of equations of Sec. 1 to the following form:

$$\begin{gathered} p = C_0\rho^{\varkappa}, \qquad \frac{v_x{}^2}{2} + \frac{\varkappa}{\varkappa - 1}\,\frac{p}{\rho} = C_1 \\ \frac{d}{dx}(h\rho v_x) = \rho_s f(p), \qquad h = h_0(x) + \int_0^t f(p)\,dt + v \\ v(x) = \eta\,\frac{\varkappa_1 + 1}{4\pi\mu} \int_{l+d}^{x} \frac{dt}{\sqrt{(l+d)^2 - t^2}} \left[-\int_{-l}^{+l} p(\tau)\,\frac{\sqrt{(l+d)^2 - \tau^2}}{\tau - t}\,d\tau \right. \end{gathered} \tag{3.2}$$

$$+\frac{\pi}{2}\sigma_y{}^\infty t+\sigma_s\int_{-l-d}^{-l}\frac{\sqrt{(l+d)^2-\tau^2}}{\tau-t}d\tau+\sigma_s\int_{l}^{l+d}\frac{\sqrt{(l+d)^2-\tau^2}}{\tau-t}d\tau\Bigg]$$

$$\int_{-l}^{+l}p(\tau)\left(\frac{l+d+\tau}{l+d-\tau}\right)^{1/2}d\tau=\frac{\pi}{2}\sigma_y{}^\infty(l+d)-\sigma_s\int_{l}^{l+d}\left(\frac{l+d+\tau}{l+d-\tau}\right)^{1/2}d\tau-\sigma_s\int_{-l-d}^{-l}\left(\frac{l+d+\tau}{l+d-\tau}\right)^{1/2}d\tau$$

$$\left(C_0=p_0\rho_0^{-\varkappa},\ \varkappa_1=3-4\nu,\ C_1=\frac{1}{2}\left(\frac{\rho_s}{\rho_1}\right)^2 f^2(p_1)+\frac{\varkappa}{\varkappa-1}\frac{p_1}{\rho_1}\right) \tag{3.2}$$

Here, μ and ν are the shear modulus and Poisson's ratio, respectively; σ_y^∞ is the working stress σ_y remote from the cavity; p_0, ρ_0 and p_1, ρ_1 are the pressure and density of the gas in the combustion chamber at $x = 0$ and at the end of the cavity at $x = l$, respectively (obviously the quantities with a subscript 1 are subject to determination); $h_0(x)$ is the given thickness of the initial cavity. The last two of Eqs. (3.2) are the result of solving the plane problem of the theory of elasticity for a cut $y = 0$, $|x| < l + d$ in an infinite plane with boundary conditions

$$\begin{aligned}
&x+iy\to\infty, && \sigma_y\to\sigma_y{}^\infty, \quad \tau_{xy}\to 0, \quad \sigma_x\to 0\\
&y=0,\quad l<|x|<l+d, && \sigma_y=\sigma_s, \quad \tau_{xy}=0\\
&y=0,\quad |x|<l && \sigma_y=-p, \quad \tau_{xy}=0
\end{aligned} \tag{3.3}$$

In order to take into account the boundary condition at the principal combustion front at $x = 0$, we introduce the correction η, equal to approximately 1.2.

System (3.2) is used to find the following quantities: d, p(x), ρ(x), v_x(x), v(x), h(x). It is easy to reduce it to a single integro-differential equation in p(x) [and the last of Eqs. (3.2) to a finite relation for d]. For a numerical solution of this equation (and only this is possible in the general case), it is most rational to employ the following method: the function p(x) is found in the form of a polynomial with unknown coefficients (for example, in the form of a linear function), and the equation is satisfied approximately in the sense of bringing the rms error as close as possible to zero. Then the unknown coefficients are determined from the condition of minimization of the function obtained. It is also possible to use Galerkin's method.

For the purpose of an estimate, we will make an approximate analytic calculation on the assumption that

$$\begin{aligned}
&h_0(x)=h_0=\text{const}, \qquad h_0\gg v, \qquad h_0\gg h_c\\
&f(p)=a+bp, \qquad \int_0^l p(x)\,dx=\frac{1}{2}l(p_1+p_0)
\end{aligned} \tag{3.4}$$

where a and b are empirical constants. In calculating the displacement v at the end of the cavity at $x = l$ and the value of d, for simplicity, we also assume that a constant pressure $1/2\ (\rho_1 + \rho_0)$ acts on the walls of the cavity.

Hence, from Eqs. (3.2) we obtain

$$\begin{aligned}
&\frac{p_1}{\rho_1{}^\varkappa}=\frac{p_0}{\rho_0{}^\varkappa}, \qquad \rho_1 v_{x1}-\rho_0 v_{x0}=\frac{\rho_s}{h_0}\left[al+\frac{1}{2}bl(p_1+p_0)\right]\\
&v_{x1}=\frac{\rho_s}{\rho^2}(a+bp_1), \qquad \frac{v_{x1}^2}{2}+\frac{\varkappa}{\varkappa-1}\frac{p_1}{\rho_1}=\frac{v_{x0}^2}{2}+\frac{\varkappa}{\varkappa-1}\frac{p_0}{\rho_0}
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&v=-\eta\frac{2(1-\nu^2)}{\pi E}l(2\sigma_s+p_1+p_0)\ln\cos\beta\\
&\frac{d}{l}=\sec\beta-1 \qquad \left(\beta=\frac{\pi(2\sigma_y^\infty+p_1+p_0)}{2(2\sigma_s+p_1+p_0)}\right)
\end{aligned} \tag{3.6}$$

Equations (3.5) are used to determine p_1, ρ_1, v_{x1}, v_{x0}. Using (2.1) and (3.6), we can find a sufficient condition of stability of the system:

$$-\eta l(2\sigma_s+p_1+p_0)\ln\cos\beta\leqslant\frac{\pi E}{2(1-\nu^2)}\delta \tag{3.7}$$

Here, ρ_1 is the root of the equation

$$\frac{\rho_s{}^2 p_0^{2/\varkappa}}{2\rho_0^2}\left(\frac{a+bp_1}{p_1^{1/\varkappa}}\right)^2+\frac{\varkappa p_0^{1/\varkappa}}{(\varkappa-1)\rho_0}p_1^{(\varkappa-1)/\varkappa}=\frac{\varkappa p_0}{(\varkappa-1)\rho_0}+\frac{\rho_s{}^2 l^2}{8h_0^2\rho_0^2}(2a+bp_1+bp_0)^2 \tag{3.8}$$

The simple solution (3.5)-(3.8) obtained can be used as the zero-order approximation.

LITERATURE CITED

1. G. P. Cherepanov, "Brittle strength of pressure vessels," PMTF, No. 6, 1969.
2. V. V. Panasyuk, Limit Analysis of Brittle Bodies with Cracks [in Russian], Naukova Dumka, Kiev, 1968.